The Berry phase for spin in the Majorana representation

1998 J. Phys. A: Math. Gen. 31 L53

(http://iopscience.iop.org/0305-4470/31/2/002)

J. Phys. A: Math. Gen. **31** (1998) L53–L59. Printed in the UK PII: S0305-4470(98)86363-4

LETTER TO THE EDITOR

The Berry phase for spin in the Majorana representation

J H Hannay
H H Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol BS8 1TL, UK

Received 28 July 1997, in final form 26 September 1997

Abstract. A continuous cyclic sequence of quantum states has an associated geometric, or Berry, phase $\mathrm{i}\oint\langle\psi|\,\mathrm{d}\psi\rangle$. For spin J, such a sequence is described by a cyclic change in the $2J+1$ coefficients a_m of the basis states $|J,m\rangle$. The Berry phase is analysed here for the general case—that is, the coefficients a_m are allowed to vary in an arbitrary cyclic manner. The result is expressed in geometric terms, specifically in the democratic representation due to Majorana. This uniquely characterizes the spin state $|\psi\rangle$, up to overall phase, by the positions of $2J$ dots on the unit sphere of directions in real space. If the positions are denoted by unit vectors $\boldsymbol{u}_k$, where $1\leqslant k\leqslant 2J$, each traces out a parametrized loop on the sphere, and the Berry phase is given by an integral of combinations of these vectors.

The geometric, or Berry phase is the real number $\mathrm{i}\oint\langle\psi|\,\mathrm{d}\psi\rangle$ associated with a cyclic sequence of quantum states $|\psi\rangle$. The situation originally envisaged by Berry [1] was that of an eigenstate $|\psi\rangle$ of a Hamiltonian being carried around by adiabatically cycling the Hamiltonian. That the Berry phase still retains some naturalness even if the cycled $|\psi\rangle$ is not an eigenstate, and even if the carrying is not adiabatic was pointed out by Aharonov and Anandan [2]. It follows from the defining expression that the Berry phase is independent of the definition of phase of the states in the cycle. A natural environment for the study of the Berry phase in a system is therefore the 'projective Hilbert space' of states irrespective of phase (the space of pure state density matrices $|\psi\rangle\langle\psi|$).

For a spin J system the usual description of a general state $|\psi\rangle$ is as a superposition of the $2J+1$ basis states with definite angular momentum component $m\hbar$ along some chosen axis: $|\psi\rangle=\sum_{m=-J}^{J}a_m|J,m\rangle$. For a continuous cyclic sequence of states $|\psi\rangle$ the corresponding set of coefficients a_m changes in a cyclic manner and the Berry phase is given by $\mathrm{i}\oint\sum_{m=-J}^{J}a_m^*\,\mathrm{d}a_m$. A well studied particular case (note (i) later) is that in which the coefficients change in such a way that the system starts and remains in a state of fixed angular momentum component $\hbar m'$ along a variable axis. The Berry phase is then minus m' times the solid angle enclosed in the circuit of axis directions. Beyond this, previous studies of the Berry phase for spin include the case of 'rigid rotation' described in note (ii), and the complete description of the Berry phase for $J=1$ mentioned in note (iii).

Here a complete description for general J is to be to be given. The set of coefficients a_m is thus allowed to perform an arbitrary cycle (subject always to normalization). The Berry phase is to be expressed in the democratic representation due to Majorana [3] in 1932. This is a natural 'projective Hilbert space' for a spin. The spin J state $|\psi\rangle$ is uniquely characterized, up to overall phase, by the positions of $2J$ dots on the unit sphere of directions in real space. Rotate the spin state and the dot pattern rotates likewise. For $J=1/2$ this is the well known Bloch sphere and the dot at position $\boldsymbol{u}$ from the centre

0305-4470/98/020053+07\$19.50

represents a 2-spinor state, call it $|\boldsymbol{u}\rangle$, which is definitely 'up' along the $\boldsymbol{u}$ direction. That is, it has density matrix $|\boldsymbol{u}\rangle\langle\boldsymbol{u}| = (1+\boldsymbol{u}\cdot\boldsymbol{\sigma})/2$, where the components of $\boldsymbol{\sigma}$ are the Pauli matrices. For higher spin values Majorana recognized the general state as the symmetrized outer product [4] of $2J$ such spin $1/2$ states with different directions vectors $\boldsymbol{u}_k$, where $1 \leqslant k \leqslant 2J$.

For a typical state (superposition of the $2J+1$ basis states) the $2J$ Majorana dots representing it are all separate [5] but for the basis states themselves the dots all lie at one or other of two antipodal axis points. The state $|J, m'\rangle$ has $(J+m')$ coincident dots along the positive axis direction and $(J-m')$ coincident dots along the negative axis direction. The extreme case, $m' = J$ when all the dots coincide is known as a spin coherent state $|'\rangle$ labelled by the axis direction. Indeed, though no use is to be made of it here, a property defining the Majorana vectors for a general state $|\psi\rangle$ is that they are the antipodes of points for which the inner product $\langle'|\psi\rangle$ has a zero as a function of axis direction [6–8].

For a cyclic sequence of states each dot $\boldsymbol{u}_k$ traces out an independent loop on the sphere (figure 1) (see also note (iv) later). The task, then, is to calculate the Berry phase in terms of these parametrized loops. It should perhaps be noted in advance that, if one is willing to sacrifice the democracy of the real space sphere, an alternative route based on stereographic projection is simpler analytically and described in note (v)

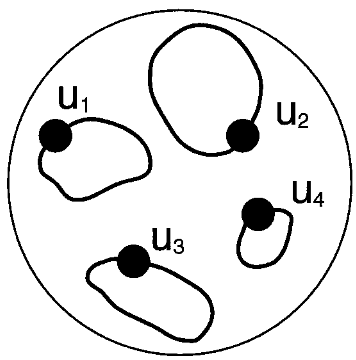

Figure 1. The Majorana representation of a cycled state of spin $J = 2$. Each of the $2J = 4$ dots traces out its own loop on the sphere.

First we recall the relevant geometrical properties of the spin $1/2$ states from which the higher spin states are to be built in the Majorana representation. The spin $1/2$ states $|\boldsymbol{u}\rangle$ and $|\boldsymbol{u}'\rangle$ described, up to overall phase, by the unit vectors $\boldsymbol{u}$ and $\boldsymbol{u}'$, have an inner product $\langle\boldsymbol{u}|\boldsymbol{u}'\rangle$ given by

$$\langle\boldsymbol{u}|\boldsymbol{u}'\rangle \equiv a\,\mathrm{e}^{\mathrm{i}A/2} \tag{1}$$

where the amplitude a is the radius of the (chord) midpoint between $\boldsymbol{u}$ and $\boldsymbol{u}'$,

$$a(\boldsymbol{u}, \boldsymbol{u}') \equiv \sqrt{\frac{1+\boldsymbol{u}\cdot\boldsymbol{u}'}{2}}. \tag{2}$$

The phase $A(\boldsymbol{u}, \boldsymbol{u}')/2$ is not determined by the vectors $\boldsymbol{u}$ and $\boldsymbol{u}'$ alone but depends on the undescribed phases of the two states. For a cyclic product of states, however, the individual phases arise in cancelling pairs. Thus, for example $[A(\boldsymbol{u}, \boldsymbol{u}') + A(\boldsymbol{u}', \boldsymbol{u}'') + A(\boldsymbol{u}'', \boldsymbol{u})]/2$, which is the phase of the product $\langle\boldsymbol{u}|\boldsymbol{u}'\rangle\langle\boldsymbol{u}'|\boldsymbol{u}''\rangle\langle\boldsymbol{u}''|\boldsymbol{u}\rangle$, *is* determined by the three vectors. Evaluating the product as $\mathrm{Tr}[(1+\boldsymbol{u}\cdot\boldsymbol{\sigma})(1+\boldsymbol{u}'\cdot\boldsymbol{\sigma})(1+\boldsymbol{u}''\cdot\boldsymbol{\sigma})]/8$ yields the complex number

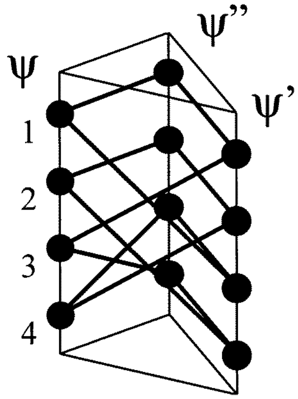

Figure 2. A schematic display of one term in the cyclic product of three states $\langle\psi|\psi'\rangle\langle\psi'|\psi''\rangle\langle\psi''|\psi\rangle$. All the dots should really lie at arbitrary positions on the unit sphere, and be connected as indicated with geodesic arcs. The J value in the picture is 2, each state having $2J = 4$ dots. Each face of the schematic prism is associated with a single factor $\langle|\rangle$. Drawn on the face is one of the 4! permutations contributing to this factor. For example, for the factor $\langle\psi'|\psi''\rangle$, the identity permutation is shown.

$x+\mathrm{i}y = [1+\boldsymbol{u}\cdot\boldsymbol{u}'+\boldsymbol{u}'\cdot\boldsymbol{u}''+\boldsymbol{u}''\cdot\boldsymbol{u}+\mathrm{i}\boldsymbol{u}\cdot(\boldsymbol{u}'\wedge\boldsymbol{u}'')]/4$. Thus $[A(\boldsymbol{u},\boldsymbol{u}')+A(\boldsymbol{u}',\boldsymbol{u}'')+A(\boldsymbol{u}'',\boldsymbol{u})]$ is twice the phase of this:

$$2\arg[1+\boldsymbol{u}\cdot\boldsymbol{u}'+\boldsymbol{u}'\cdot\boldsymbol{u}''+\boldsymbol{u}''\cdot\boldsymbol{u}+\mathrm{i}\boldsymbol{u}\cdot(\boldsymbol{u}'\wedge\boldsymbol{u}'')]. \tag{3}$$

It is ambiguous (by π) to write the arg as $\arctan(y/x)$ though it could be written as $2\arctan(y/(x+\sqrt{x^2+y^2}))$. The quantity (3) is recognized (following the remarkable work of Panchatnam [9, 10] in optics) as the area or solid angle of the spherical triangle $(\boldsymbol{u},\boldsymbol{u}',\boldsymbol{u}'')$. Particularly useful below will be the area of an infinitely thin triangle in which $\boldsymbol{u}''=\boldsymbol{u}'+\mathrm{d}\boldsymbol{u}'$, for which (3) yields

$$\frac{\mathrm{d}\boldsymbol{u}'\cdot(\boldsymbol{u}\wedge\boldsymbol{u}')}{1+\boldsymbol{u}\cdot\boldsymbol{u}'}. \tag{4}$$

Similarly for any cyclic sequence $A(\boldsymbol{u},\boldsymbol{u}')+A(\boldsymbol{u}',\boldsymbol{u}'')+\cdots+A(\boldsymbol{u}''^{\cdots\prime},\boldsymbol{u})$ is the area or solid angle of the spherical polygon $(\boldsymbol{u},\boldsymbol{u}',\boldsymbol{u}'',\ldots,\boldsymbol{u}''^{\cdots\prime})\ (\mathrm{mod}\,4\pi)$.

A general state $|\psi\rangle$ of spin J is the normalized symmetrized outer product of spin 1/2 states: $|\psi\rangle = |\{\boldsymbol{u}_k\}\rangle/(\langle\{\boldsymbol{u}_k\}|\{\boldsymbol{u}_k\}\rangle)^{1/2}$, where

$$|\{\boldsymbol{u}_k\}\rangle = ((2J)!)^{-1/2}\sum_P |\boldsymbol{u}_{P1}\rangle\times|\boldsymbol{u}_{P2}\rangle\times\cdots\times|\boldsymbol{u}_{P2J}\rangle \tag{5}$$

the sum being over all $(2J)!$ permutations P, taking $1,2,3,\ldots,2J$ to $P1$, $P2$, $P3,\ldots,P2J$.

The inner product $\langle\psi|\psi'\rangle$ of two states $|\psi\rangle$ and $|\psi'\rangle$ is then

$$\langle\psi|\psi'\rangle = \frac{\langle\{\boldsymbol{u}_k\}|\{\boldsymbol{u}'_k\}\rangle}{\sqrt{\langle\{\boldsymbol{u}_k\}|\{\boldsymbol{u}_k\}\rangle}\sqrt{\langle\{\boldsymbol{u}'_k\}|\{\boldsymbol{u}'_k\}\rangle}} \tag{6}$$

where

$$\langle\{\boldsymbol{u}_k\}|\{\boldsymbol{u}'_k\}\rangle = \sum_P\prod_k\langle\boldsymbol{u}_k|\boldsymbol{u}'_{Pk}\rangle \tag{7}$$

$$=\sum_P\prod_k a(\boldsymbol{u}_k,\boldsymbol{u}'_{Pk})\exp\{\mathrm{i}A(\boldsymbol{u}_k,\boldsymbol{u}'_{Pk})/2\} \tag{8}$$

the products being over the $2J$ values $1 \leqslant k \leqslant 2J$. The numerator (7) still depends on the individual phases A of the states. Again, however, if a state jumps through a discrete cycle $|\psi\rangle \to |\psi'\rangle \to |\psi''\rangle \to |\psi\rangle$, and the cyclic product $\langle\psi|\psi'\rangle\langle\psi'|\psi''\rangle\langle\psi''|\psi\rangle$ is formed, the individual phases cancel. The mechanism by which they do so is that the product of the three permutations in each term of (7) is associated (figure 2) with one or more *closed* spherical polygons (with a total of $3 \times 2J$ sides). The phase of each product term is equal to half its total polygon area.

The Berry phase ϕ of a continuous cycle of states, $|\psi\rangle \to |\psi'\rangle \to \cdots |\psi'^{\cdots\prime}\rangle \to |\psi\rangle$, is given by $\mathrm{e}^{-\mathrm{i}\phi} = \langle\psi|\psi'\rangle\langle\psi'|\psi''\rangle \cdots \langle\psi'^{\cdots\prime}|\psi\rangle$, where there are infinitely many, infinitesimally different, states whose inner products therefore have unit modulus (to the required first order). To evaluate it, it is only necessary to examine an individual numerator term of the type (7), with $\boldsymbol{u}'_k = \boldsymbol{u}_k + \mathrm{d}\boldsymbol{u}_k$. With a notation to be explained,

$$\langle\{\boldsymbol{u}_k\}|\{\boldsymbol{u}_k + \mathrm{d}\boldsymbol{u}_k\}\rangle = \sum_P \prod_k a(\boldsymbol{u}_k, \boldsymbol{u}_{Pk} + \mathrm{d}\boldsymbol{u}_{Pk}) \exp\{\mathrm{i}A(\boldsymbol{u}_k, \boldsymbol{u}_{Pk} + \mathrm{d}\boldsymbol{u}_{Pk})/2\} \tag{9}$$

$$\equiv \sum_P (a_P + \Delta a_P)\, \mathrm{e}^{\mathrm{i}(A_P + \Delta A_P)/2} \tag{10}$$

$$= \mathrm{e}^{\mathrm{i}\Delta A_I/2} \sum_P (a_P + \Delta a_P) \mathrm{e}^{\mathrm{i}(A_P + \Delta A_P - \Delta A_I)/2} \tag{11}$$

$$\approx \mathrm{e}^{\mathrm{i}\Delta A_I/2} \left(\sum_P a_P \mathrm{e}^{\mathrm{i}A_P/2} + \sum_P \Delta a_P \mathrm{e}^{\mathrm{i}A_P/2} + \mathrm{i} \sum_P \frac{1}{2} (\Delta A_P - \Delta A_I) a_P \mathrm{e}^{\mathrm{i}A_P/2} \right) \tag{12}$$

$$\approx \mathrm{e}^{\mathrm{i}\Delta A_I/2} \left(\sum_P a_P \cos(A_p/2) + \mathrm{i} \sum_P [\Delta a_P \sin(A_P/2) + \frac{1}{2} (\Delta A_P - \Delta A_I) a_P \cos(A_P/2)] \right). \tag{13}$$

Here, apart from introducing the notation defined below and discarding all but leading order real and imaginary terms, the only operation was recognizing that the first sum in (12) is real because $a_P = a_{P^{-1}}$, and $A_P = -A_{P^{-1}}$ where P^{-1} is the inverse of permutation P. The notation is

$$a_P \equiv \prod_k a(\boldsymbol{u}_k, \boldsymbol{u}_{Pk}) = \prod_k \sqrt{\frac{1 + \boldsymbol{u}_k \cdot \boldsymbol{u}_{Pk}}{2}} \tag{14}$$

$$\Delta a_P \equiv a_P \sum_k \frac{1}{2} \frac{\boldsymbol{u}_k \cdot \mathrm{d}\boldsymbol{u}_{Pk}}{(1 + \boldsymbol{u}_k \cdot \boldsymbol{u}_{Pk})} \tag{15}$$

$$\Delta A_P - \Delta A_I \equiv \sum_k \frac{\boldsymbol{u}_k \cdot (\mathrm{d}\boldsymbol{u}_{Pk} \wedge \boldsymbol{u}_{Pk})}{(1 + \boldsymbol{u}_k \cdot \boldsymbol{u}_{Pk})} \tag{16}$$

where the summand here is, from (4), the area of the spherical triangle $(\boldsymbol{u}_k, \boldsymbol{u}_{Pk}, \boldsymbol{u}_{Pk} + \mathrm{d}\boldsymbol{u}_{Pk})$ (figure 3).

A_P is the total area of the polygons on the sphere generated by the $2J$ geodesic arcs $\boldsymbol{u}_k \to \boldsymbol{u}_{Pk}$. This can be expressed as the sum of areas (3) of the spherical triangles $(\boldsymbol{n}, \boldsymbol{u}_k, \boldsymbol{u}_{Pk})$, where $\boldsymbol{n}$ is any unit vector (for example, it could be taken as one of the $\boldsymbol{u}_k$).

$$A_P = \sum_k 2\arg[(1 + \boldsymbol{n} \cdot \boldsymbol{u}_k + \boldsymbol{u}_k \cdot \boldsymbol{u}_{Pk} + \boldsymbol{u}_{Pk} \cdot \boldsymbol{n}) + \mathrm{i}\boldsymbol{n} \cdot (\boldsymbol{u}_k \wedge \boldsymbol{u}_{Pk})]. \tag{17}$$

Finally, there is the quantity ΔA_I. It is twice the phase factor associated with the identity permutation. It is not necessary, or possible, to supply ΔA_I explicitly (it depends on the

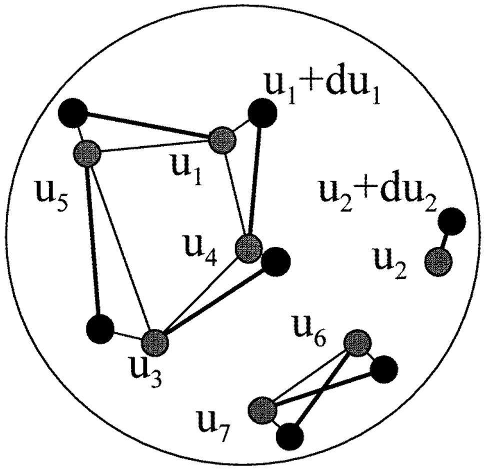

Figure 3. The Majorana sphere for a cycling state with $J = 7/2$ so that there are seven dots each with a small displacement (the remainder of the cycle is not shown). This whole picture represents the contribution of a particular permutation $(1, 2, 3, 4, 5, 6, 7) \to (5, 2, 4, 1, 3, 7, 6)$, to $\langle\psi|\psi + \mathrm{d}\psi\rangle$, and is analogous to a single face of the prism in figure 2. Among the quantities required is the phase change as each geodesic arc swings from its old to its new position (feint to bold). This equals half the area of the thin triangle swept plus the phase change along the short leg of the triangle $\mathrm{d}\boldsymbol{u}$. The sum of the seven arc swing phase changes is denoted $\Delta A_P/2$.

individual phases of the states). All that will be required is the integral, A_I, of ΔA_I, which is simply the total area of the loops traced out by the $\boldsymbol{u}_k$. An explicit expression for this total area as an integral of thin triangle areas (4) is, with $\boldsymbol{n}$ any fixed unit vector,

$$A_I \equiv \oint \Delta A_I = \sum_k \oint \frac{\mathrm{d}\boldsymbol{u}_k \cdot (\boldsymbol{n} \wedge \boldsymbol{u}_k)}{(1 + \boldsymbol{u}_k \cdot \boldsymbol{n})}. \tag{18}$$

The Berry phase ϕ is read off as the phase of (13), using (15) and (16):

$$\begin{aligned}\phi = -\frac{1}{2}A_I - \oint\Bigg\{ \sum_P \Bigg[\frac{1}{2}a_P \sin(A_P/2)\left(\sum_k \frac{\boldsymbol{u}_k \cdot \mathrm{d}\boldsymbol{u}_{Pk}}{1 + \boldsymbol{u}_k \cdot \boldsymbol{u}_{Pk}}\right) \\ + \frac{1}{2}a_P \cos(A_P/2)\left(\sum_k \frac{\boldsymbol{u}_k \cdot (\mathrm{d}\boldsymbol{u}_{Pk} \wedge \boldsymbol{u}_{Pk})}{1 + \boldsymbol{u}_k \cdot \boldsymbol{u}_{Pk}}\right)\Bigg]\Bigg\}\left(\sum_P a_P \cos(A_P/2)\right)^{-1}\end{aligned} \tag{19}$$

which, with the definitions (14), (17) and (18), is the result. There follow several notes.

(i) The usual special case in which the spin axis is cycled with constant axial angular momentum component $\hbar m'$ is easily extracted from (19). The state $|J, m'\rangle$ has $(J + m')$ dots in the positive axis direction and $(J - m')$ in the negative axis direction. The vector cross products render all but the first term of (19) zero. Thus the Berry phase is just minus half the total area of the cycles of the dots: $[(J + m') - (J - m')]/2 = m'$ times the area of the axis direction cycle, as expected.

(ii) A more general special case is that of an arbitrary initial state subject to a time-dependent Hamiltonian *linear* in the angular momentum operators. This produces a 'rigid rotation' of the state back to itself with a variable angular velocity vector $\boldsymbol{\omega}(t)$ (normally induced physically by a magnetic field $\boldsymbol{B}(t)$). The Berry phase for this case (or special cases of it) have also been studied previously [11–13], though the simplicity of the result, (20), from the Majorana analysis, has not emerged. In fact, following discussions with Berry (20) can be understood so straightforwardly (see the end of this note), that all derivations, including this, are really redundant. Nonetheless, the Majorana derivation is instructive, yielding (21) as a corollary, and is given after the statement of the result.

$$\phi = 2\pi n J + \int \boldsymbol{\omega}(t)\cdot\boldsymbol{S}(t)\,\mathrm{d}t. \tag{20}$$

Here $\hbar\boldsymbol{S}$ is the spin angular momentum vector $\hbar(\langle S_x\rangle,\ \langle S_y\rangle,\ \langle S_z\rangle)$. This vector obeys $\mathrm{d}\boldsymbol{S}/\mathrm{d}t = \boldsymbol{\omega}\wedge\boldsymbol{S}$ and has constant magnitude. The integer n is the number of complete turns of the spin. Only its value mod 2 is required (since $2J$ is an integer and only $\phi \bmod 2\pi$ is meaningful) and only its value mod 2 is well defined. It is zero or one depending on whether the cycle of orientations passed through in the rigid rotation is 'contractible' or not in the space of orientations, SO(3).

The derivation of (20) from (19) follows from the fact [14] that, since the dot pattern rotates rigidly, the cycle area of the dot $\boldsymbol{u}_k$ is expressible geometrically as $2\pi n - \int \boldsymbol{\omega}(t)\cdot\boldsymbol{u}_k(t)\,\mathrm{d}t \pmod{4\pi}$. Together with the substitution $\mathrm{d}\boldsymbol{u}_k = \boldsymbol{\omega}\wedge\boldsymbol{u}_k\,\mathrm{d}t$ in the integral in (19), this gives the Berry phase in the form $2\pi nJ + \int \boldsymbol{\omega}(t)\cdot\boldsymbol{v}(t)\,\mathrm{d}t$ where the vector $\boldsymbol{v}$, depending only on the set $\boldsymbol{u}_k$, moves rigidly with them (i.e. $\mathrm{d}\boldsymbol{v}/\mathrm{d}t = \boldsymbol{\omega}\wedge\boldsymbol{v}$). To identify $\boldsymbol{v}$ with $\boldsymbol{S}$ it is only necessary to examine the case of constant $\boldsymbol{\omega}$ for which case the Berry phase is found independently as the total phase $2\pi nJ$, minus the dynamical phase $-\boldsymbol{S}\cdot\boldsymbol{\omega}t$ with $|\boldsymbol{\omega}|t = 2\pi n$ for n complete turns.

An incidental benefit of this last identification is an explicit formula for $\boldsymbol{S}$:

$$\begin{aligned}\boldsymbol{S} = \frac{1}{2}\sum_k \boldsymbol{u}_k - \Bigg\{ \sum_P \Bigg[\frac{1}{2} a_P \sin(A_P/2) \Bigg(\sum_k \frac{\boldsymbol{u}_{Pk}\wedge\boldsymbol{u}_k}{1+\boldsymbol{u}_k\cdot\boldsymbol{u}_{Pk}} \Bigg) \\ - \frac{1}{2} a_P \cos(A_P/2) \Bigg(\sum_k \frac{\boldsymbol{u}_{Pk}\wedge(\boldsymbol{u}_k\wedge\boldsymbol{u}_k)}{1+\boldsymbol{u}_k\cdot\boldsymbol{u}_{Pk}} \Bigg) \Bigg] \Bigg\} \Bigg(\sum_P a_P \cos(A_P/2) \Bigg)^{-1}.\end{aligned} \tag{21}$$

To make the result (20) obvious it can be interpreted as total phase minus dynamical phase (in the Aharonov–Anandan sense). The total phase comes from the spin J representation of the unitary operator effecting the whole rotation, namely the identity operator times $\exp(\mathrm{i}2\pi nJ)$.

(iii) For the case $J = 1$ the general Berry phase was found previously by Bouchiat and Gibbons [15] by a quite different method. In the present description the result (19) simplifies considerably since there are only two dots, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, and the area A_P is zero

$$\phi = -\oint \frac{1}{2}\Delta A_I - \frac{1}{2}\oint \frac{(\mathrm{d}\boldsymbol{u}_1 - \mathrm{d}\boldsymbol{u}_2)\cdot(\boldsymbol{u}_1\wedge\boldsymbol{u}_2)}{3+\boldsymbol{u}_1\cdot\boldsymbol{u}_2}. \tag{22}$$

With some manipulation this accords with the formula of Bouchiat and Gibbons. Since photons have spin 1 it applies to optics [16].

(iv) Instead of each of the dots $\boldsymbol{u}_k$ moving around its own separate cycle, they can permute: each can move from its own initial position to the initial position of another. Since the dots are not distinguished, the final state of the system is the same as the initial state, as is required for the definition of the Berry phase. Together, the paths of the dots form one or more closed loops. The result (19) is unchanged (as is (25) below), except that the loop integral symbols should strictly be replaced by ordinary integrals since the individual vectors $\boldsymbol{u}_k$ do not complete loops.

An interesting instance of this alternative is when the initial state has an especially symmetrical pattern of dots, for example two sets of J coincident dots at antipodal points (for integer J). This state, when subject to rigid rotations, so that the dots remain antipodal, has zero Berry phase for any cycle in which the antipodes do not exchange positions, but a phase factor of $(-1)^J$ if they exchange. This fact, which follows directly from (19), with only the first term non-zero, is the interpretation, in the Majorana representation, of the '$m = 0$' theorem of Robbins and Berry [17].

(v) A more compact formulation of the Berry phase for spin J derives from representing the dots $\boldsymbol{u}_k$ not on the unit sphere but in stereographic projection from a chosen 'south pole' $-\boldsymbol{n}$ onto the equatorial plane (x, y). Indeed this device due to Majorana himself is the usual framework for calculations in his representation. Thus the vectors $\boldsymbol{u}_k$ map to the points $(x_k, y_k) \equiv (\boldsymbol{u}_k \cdot \hat{\boldsymbol{x}}/[1+\boldsymbol{u}_k \cdot \boldsymbol{n}], \boldsymbol{u}_k \cdot \hat{\boldsymbol{y}}/[1+\boldsymbol{u}_k \cdot \boldsymbol{n}])$. Combining the coordinates in this plane into a complex number $z_k = x_k + \mathrm{i}y_k$, the inner product of two unnormalized states is

$$\langle\{\boldsymbol{u}_k\}|\{\boldsymbol{u}'_k\}\rangle = \sum_P \prod_k [(1 + z_k^* z'_{Pk})/\sqrt{1 + z_k^* z_k}\sqrt{1 + z_k'^* z'_k}]. \tag{23}$$

The square root products can be taken out of the sum and are thus merely further normalization which is irrelevant for the phase. Substituting $z'_k = z_k + \mathrm{d}z_k$, the remaining sum becomes

$$\sum_P \prod_k (1 + z_k^* z_{Pk} + z_k^* \,\mathrm{d}z_{Pk}) \approx \sum_P \left(\left[\prod_k (1 + z_k^* z_{Pk}) \right] \left[1 + \sum_k \frac{z_k^* \,\mathrm{d}z_{Pk}}{1 + z_k^* z_{Pk}} \right] \right) \tag{24}$$

from which the Berry phase can be read off:

$$\phi = -\oint \frac{\sum_P \mathrm{Im} \left[\left(\prod_k (1 + z_k^* z_{Pk}) \right) \sum_k \frac{z_k^* \,\mathrm{d}z_{Pk}}{1 + z_k^* z_{Pk}} \right]}{\sum_P \left[\prod_k (1 + z_k^* z_{Pk}) \right]}. \tag{25}$$

References

[1] Berry M V 1984 *Proc. R. Soc.* A **392** 45–57
[2] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593–6
[3] Majorana E 1932 *Nuovo Cimento* **9** 43–50
[4] Penrose R and Rindler W 1984 *Spinors and Spacetime* vol 1 (Cambridge: Cambridge University Press) p 162
[5] Hannay J H 1996 *J. Phys. A: Math. Gen.* **29** L101–5
[6] Leboeuf P 1991 *J. Phys. A: Math. Gen.* **24** 4575–86
[7] Penrose R 1989 *The Emperor's New Mind* (Oxford: Oxford University Press)
[8] Penrose R 1994 *Shadows of the Mind* (Oxford: Oxford University Press) figure 5.21
[9] Pancharatnam S 1956 *Proc. Ind. Acad. Sci.* A **44** 247–62
[10] Berry M V 1987 *J. Mod. Opt.* **34** 1401–7
[11] Layton E 1990 *Phys. Rev.* A **41** 42–8
[12] Gao X, Xu J and Qian T 1991 *Phys. Lett.* **152A** 449–52
[13] Skrynnikov N R, Zhou J and Sanctuary B C 1994 *J. Phys. A: Math. Gen.* **27** 6253–65
[14] Hannay J H 1998 *J. Phys. A: Math. Gen.* **31** submitted
[15] Bouchiat C and Gibbons G W 1988 *J. Physique* **49** 187–99
[16] Hannay J H 1998 *J. Mod. Opt.* to appear
[17] Robbins J M and Berry M V 1994 *J. Phys. A: Math. Gen.* **27** L435–8